ON THE EXPECTED OUTPUT ANALYSIS OF TWO-STAGE TRANSFER-LINE PRODUCTION SYSTEMS SUBJECT TO INSPECTIONS AND REWORK[1]

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This paper deals with the transient analysis of two-stage transfer-line production systems subject to an initial storage of unlimited capacity and inter-stage, end-stage inspections. It provides an integrated framework to consider manufacturing, inspection and rework activities simultaneously. Rework of a defective item produced by a machine is done on the same machine itself. Explicit expressions for some of the system characteristics have been obtained using the state-space method and regeneration point technique. All the random variables involved in the analysis are assumed to be arbitrarily distributed (i.e., general).

Introduction

Consider the following problem in a two-stage transfer-line production system. Products coming out of machine I are inspected at an inspection point before it is being transferred to machine II for further processing. While the good ones are transferred to machine II, the products that are not confirming to specifications are further classified as products that are reworkable and otherwise. In the latter case the product is scrapped. A similar strategy is adopted for the products coming out of machine II.

The main reason to study productive systems is that every enterprise, private or public, manufacturing or service, involves a productive system. There is an operation function in all enterprises. In manufacturing, the productive system is of great importance within the enterprise as a whole. In many service organizations, the produtive system and the product offered are so completely bound up together that they are indistinguishable. While the 'production line' plays an important role in our life, very little research on the interactions between the stages in a line is reported [1].

The focus of analysis of this paper is discrete part manufacturing systems, where each item processed is distinct. Such systems are normal in mechanical,

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electrical and electronics industries making products such as cars, refrigerators, electric generators, or computers. The analysis of production systems, though not given importance to the extent it deserves, is one of the oldest problems in industrial engineering [2].

Variation in the production rate of the stations may be due to external causes such as power supply failures, material shortages, strikes, or perhaps the way incoming orders arrive and production plans are prepared [3].

The efficiency of a transfer line with no inventory banks can be substantially less than that of the efficiency of the individual stages. Inventory banks provide a means of improving the line efficiency so that it becomes closer to the efficiency of the worst stage, that is, the stage with the lowest throughput if it were operated on its own.

Systems without internal storages are frequently encountered in industry. In that case, since there is no buffer in between machines, the behaviour of each machine is highly dependent on one another due to the effect of blocking. Two types of behaviour are encountered in such lines: synchronous behaviour and asynchronous behaviour. In the case of asynchronous behaviour, parts can move independently of each other, whereas in the case of synchronous behaviour, transfer of parts from one machine to the next one occur simultaneously. This may be the case, for instance, when a rigid parts transfer system is used. It should however be noticed that in the case of two-machine transfer lines, it is easy to show that the production rate obtained using asynchronous transfer is greater than that corresponding to a synchronous transfer.

Thus, the production rate of a transfer line with synchronous behaviour provides a lower bound on the production rate of the same line with asynchronous behaviour.

The basic causes of problems in production lines are different production rates, variability of the service times due to randomness, and station breakdowns. Losses in line efficiency are evidenced in periods where a station is blocked or starved. A station is blocked if the service of the item in this station is completed and service in the next station is still going on so that it is not possible for the item to enter the next station. In this case, the station remains idle until the service in the next station is completed. A station is starved if there are no items either in the buffer or in service.

The analysis of the two-stage systems provides useful hints to describe generalised (i.e., n-stage) systems and such an analysis is getting a great deal of attention over the last few years. This is because of the reason that any multistage system can be analysed by formulating the system as a two-stage system [4].

Several authors [5,. . .,16] have analysed production systems to find various measures of system performance. But, in all their works, inspection has not been taken into account or rejected items were scrapped. But, this may not be feasible always. This is particularly so when the cost of an item is high. In fact, as it has been pointed out by Gupta and Chakraborty [17], rework is becoming inevitable in many production systems. Not much work has been reported on rework. Some authors have suggested the rework of rejected items, but their analysis is confined to deterministic models [18]. Others have considered only Markovian approach. Also, most of the work in the literature are mainly concentrated on the analysis of steady-state behaviour of the system which may not be useful in reality, as most of the systems will breakdown or collapse before reaching steady-state. Some authors [19,20,21] have analysed the transient behaviour of the system without taking into account the concept of rework, and the models are too specific as they deal with Markovian distributions only.

The present paper deals with the transient analysis of a family of two-stage production systems (since, the processing times at both the stages including rework are assumed to be arbitrarily distributed) subject to an initial buffer of unlimited capacity, inter-stage, end-stage inspections. The processing time of each type of rework is governed by a different distribution. The analysis is carried out by modelling the production system as a queueing system.

Production system under study is modelled using regeneration point technique. For details of this approach, we refer for Uematsu et al [22], Birolini [23]. Integral equations have been written for various state probabilities by identifying the system at suitable regeneration epochs [24]. These equations, which are of convolution type, have been solved by successive approximation [25] .

The following system characteristics have been obtained under the assumption that the distributions of all the random variables involved in the analysis are arbitrary.

1. Expected number of jobs completed by machine I in $[0, t]$.
2. Expected number of jobs completed by machine II in $[0, t]$.
3. Expected number of visits of the system to blocked state in $[0, t]$.
4. Expected number of reworked jobs completed by machine I in $[0, t]$.
5. Expected number of reworked jobs completed by machine II in $[0, t]$.
6. Expected number of visits of the system to rework state (i.e., to a state in which either machine I or machine II is busy with rework) in $[0, t]$.

The presentation of contents of this paper is organised as follows. Section 2 gives a list of assumptions we made, section 3 gives notations used, while sections 4 and 5 deal with system modelling and evaluation of system characteristics respectively. Numerical illustrations are given for some particular cases in section 6. Section 7 is devoted to conclusion.

2. Assumptions

1. Initial buffer is of unlimited capacity and hence machine I is never starved.
2. Transfer of units from the initial buffer to machine I and from machine I to machine II are instantaneous.
3. Inspections at both the inter-stage and end-stage are of instantaneous type.
4. Whenever a product is to be reworked, then the respective machine will immediately start reworking the defective product.
5. Processing times at both the stages are independent, random and arbitrarily distributed (including processing times of rework).
6. Products from machine I will be inspected ony when machine II is free. (This is to avoid indefinite blocking.)
7. Stage II (i.e., machine II) is never blocked.
8. Reworked jobs are always perfect.
9. Machine I/II is perfect (i.e., reliable).
10. Setup is instantaneous.

3. Notations

pdf	:	probability density function
cdf	:	cumulative distribution fucntion
sf	:	survivor function
$f1(\cdot)/f2(\cdot)$	:	pdf of processing time of machine I/machine II
$F1(\cdot)/F2(\cdot)$	:	cdf of processing time of machine I/II
$\overline{F}1(\cdot)/\overline{F}2(\cdot)$	:	sf of processing time of machine I/II
$g_1(\cdot)/g_2(\cdot)$	:	pdf of processing time of rework in machine I/II
$G_1(\cdot)/G_2(\cdot)$	:	cdf of processing time of rework in machine I/II
$\overline{G}_1(\cdot)/\overline{G}_2(\cdot)$	:	sf of processing time of rework in machine I/II
p_{g1}/p_{g2}	:	probability of a job completed by machine I/II is good
p_{r1}/p_{r2}	:	probability of a job completed by machine I/II is not good but reworkable
p_{s1}/p_{s2}	:	probability of a job completed by machine I/II is neither good nor reworkable and hence a scrap.
$*$	:	convolution: $f(t) * g(t) = \int_0^t f(u)g(t-u)du$

4. System modelling

The system under consideration is modelled by identifying the state of the system at any instant t. The possible states of the system are given in *Table 1*. *Figure 1* represents the schematic diagram of the production system. All the one-step transitions are searched between states 1 and 6 and are depicted in *Figure 2*.

Table 1: State space

State	Machine I	Machine II
1	Busy	Free
2	Busy with rework	Free
3	Busy	Busy
4	Busy	Busy with rework
5	Blocked	Busy
6	Blocked	Busy with rework

5. Evaluation of system characteristics

In this Section, expressions for the various measures of system characteristics have been obtained.

5.1 Expected number of jobs completed by machine I in [0, t]

The expression for the expected number of jobs completed by machine I in $[0,t]$ is obtained as follows.

Let $M_1^I(t)$ denote the expected number of jobs completed by machine I in $(0,t]$, given that the system was in state 1 at time $t=0$.

Starting with state 1, the next regenerative transition is to state 3 (i.e., the product from machine I is good) or, to state 2 (i.e., the product from machine I is not good but reworkable), or to state 1 itself (i.e., the product from machine I is neither good nor reworkable), with probabilities p_{g1}, p_{r1} and p_{s1} respectively; i.e.,

$$M_1^I(t) = f_1(t) * \left[p_{g1} M_3^I(t) + p_{r1} M_2^I(t) + p_{s1}^I M_1^I(t)\right] + F_1(t)\,. \tag{1}$$

Following a similar logic, one can obtain the remaining equations.

$$M_2^I(t) = g_1(t) * \left[M_3^I(t)\right] + G_1(t) \tag{2}$$

$$\begin{aligned}
M_3^I(t) = \quad & [f_1(t)F_2(t) + f_2(t)F_1(t)] * [p_{g1}(p_{g2}+p_{s2})M_3^I(t) \\
& + p_{r1}(p_{g2}+p_{s2})M_2^I(t) + p_{s1}(p_{g2}+p_{s2})M_1^I(t)] \\
& + p_{r2}\left[F_1(t)f_2(t)\right] * M_6^I(t) \\
& + p_{r2}\int_0^t \left[\int_0^u f_2(u_1)g_2(u-u_1)\int_{u_1}^u f_1(v)\,dv\,du_1 \right. \\
& \left. + f_1(u)\int_0^u f_2(u_1)G_2(u-u_1)du_1\right] \\
& \qquad [p_{g1}M_3^I(t-u) + p_{s1}M_1^I(t-u) + p_{r1}M_2^I(t-u)]\,du \\
& + (p_{g2}+p_{s2})\int_0^t [f_1(u)F_2(u) + f_2(u)F_1(u)]\,du \\
& + p_{r2}\int_0^t \left[\int_0^u f_2(u_1)G_2(u-u_1)\int_{u_1}^u f_1(v)dvdu_1 \right. \\
& \left. + f_1(u)\int_0^u f_2(u_1)G_2(u-u_1)du_1\right] du + p_{r2}\int_0^t f_2(u)F_1(u)du
\end{aligned} \tag{3}$$

$$M_6^I(t) = g_2(t) * \left[p_{g1}M_3^I(t) + p_{r1}M_2^I(t) + p_{s1}M_1^I(t)\right]\,. \tag{4}$$

The above set of integral equations can be arranged in matrix form (refer to

Jones [25]) as follows:

$$\mathbf{G}(t) - \int_0^t \mathbf{W}(u)\mathbf{G}(t-u)du = \mathbf{L}(t)$$

where $\mathbf{W}$ is a square matrix of order n (n = the number of equations) consisting of the coefficients of M_i^I's, $\mathbf{G}$ and $\mathbf{L}$ are column matrices of order $n \times 1$ consisting of M_i^I's and terms independent of M_i^I's respectively.

The above set of integral equations, being of convolution type, can be solved by the method suggested by Jones [25].

5.2 Expected number of jobs completed by machine II in $[0, t]$

Let $M_1^{II}(t)$ denote the expected number of jobs completed by machine II in $[0, t]$, given that the system was in state 1 at time $t = 0$.

The set of equations corresponding to this case can be obtained using the logic similar to the one given in the previous Subsection.

The matrices $\mathbf{G}$ and $\mathbf{W}$ will remain the same, with terms M_i^I's being replaced by M_i^{II}'s, whereas the $\mathbf{L}$ matrix will be of the form

$$\mathbf{L} = [L_1, L_2, L_3, L_4]^T$$

where

$$L_1 = L_2 = 0 ,$$

$$L_3 = (p_{g2} + p_{s2}) \int_0^t [f_1(u)F_2(u) + f_2(u)F_1(u)]du +$$
$$p_{r2} \int_0^t \left[\int_0^u f_2(u_1)g_2(u-u_1) \int_{u_1}^u f_1(v)dvdu_1 + \right.$$
$$\left. f_1(u) \int_0^u f_2(u_1)G_2(u-u_1)du_1 \right] du +$$
$$p_{r2} \int_0^t f_2(u)F_1(u)du ,$$
$$L_4 = G_2(t) .$$

5.3 Expected number of visits of the system to blocked state in $[0, t]$

Let $M_1^{BL}(t)$ denote the expected number of visits of the system to blocked state in $[0, t]$, given that the system was in state 1 at time $t = 0$.

The L_i elements of **L** matrix, corresponding to this case, are

$$L_1 = L_2 = 0 \ ,$$

$$L_3 = \int_0^t f_2(u)F_1(u)du +$$

$$p_{r2} \int_0^t \left[\int_0^u f_2(u_1)g_2(u-u_1) \int_{u_1}^u f_1(v)dvdu_1 \right] du \ ,$$

$$L_4 = 0$$

5.4 Expected number of reworked jobs completed by machine I in [0, t]

Let $M_1^{R1}(t)$ denote the expected number of jobs completed by machine I in $[0,t]$, given that the system was in state 1 at time $t = 0$. The L_i elements of **L** matrix, corresponding to this case, are:

$$L_1 = 0 \ , \qquad L_2 = G_1(t) \ , \qquad L_3 = L_4 = 0 \ .$$

5.5 Expected number of reworked jobs completed by machine II in [0, t]

Let $M_1^{R2}(t)$ denote the expected number of reworked jobs completed by machine II in $[0,t]$, given that the system was in state 1 at time $t = 0$. The L_i elements of **L** matrix, corresponding to this case, are

$$L_1 = L_2 = 0 \ ,$$

$$L_3 = p_{r2} \int_0^t \left[\int_0^u f_2(u_1)g_2(u-u_1) \int_{u_1}^u f_1(v)dvdu_1 + \right.$$

$$\left. f_1(u) \int_0^u f_2(u_1)G_2(u-u_1)du_1 \right] du];,$$

$$L_4 = G_2(t) \ .$$

5.6 Expected number of visits of the system to rework state in [0, t]

Let $M_1^{RS}(t)$ denote the expected number of visits of the system to rework state in $[0,t]$, given that the system was in state 1 at time $t = 0$. The L_i elements of **L** matrix, corresponding to this case, are

$$L_1 = p_{r1}F_1(t) \ , \qquad L_2 = 0 \ ,$$

$$L_3 = p_{r2} \int_0^t [f_2(u)F_1(u)]\,du +$$

$$p_{r1}(p_{g2} + p_{s2}) \int_0^t [f_1(u)F_2(u) + f_2(u)F_1(u)]\,du +$$

$$p_{r1}p_{r2} \int_0^t \left[\int_0^u f_2(u_1)g_2(u-u_1) \int_{u_1}^u f_1(v)\,dv\,du_1 + \right.$$

$$\left. f_1(u) \int_0^u f_2(u_1)G_2(u-u_1)\,du_1 \right] du\,,$$

$$L_4 = p_{r1}\left[G_2(t)\right] .$$

6. Numerical illustrations

Programs have been devised to obtain the numerical values. The numerical values for the expected number of jobs completed by machine I; machine II, reworked jobs completed by machine I in the interval $[0, t]$ are given in Tables 2 and 4. The numerical values for the expected number of reworked jobs completed by machine II, expected number of visits of the system to blocked state, rework state in $[0, t]$ are given in Tables 3 and 5 for some selected values of parameters where

$$f_1(t) = \lambda_1 \exp(-\lambda_1 t)\,, \qquad f_2(t) = \lambda_2^2 t \exp(-\lambda_2 t)\,,$$

$$g_1(t) = \lambda_3 t^{\lambda_3 - 1} \exp(-t^{\lambda_3})\,, \qquad g_2(t) = \lambda_4 \exp(-\lambda_4 t)\,.$$

Sensitivity of the numerical values with respect to changes in parameters are obvious from the Tables.

7. Conclusion

In this paper, the concept of rework is incorporated in the probabilistic analysis of two-stage transfer-line production systems with an initial storage of unlimited capacity. A stochastic model of a two-stage production system subject to an initial buffer of unlimited capacity, inter-stage and end-stage inspections and rework is developed by modelling the production system as a queueing system. Analytical expressions for some of the measures of system performance such as expected number of jobs completed by machines I/II, expected number of reworked jobs completed by machines I, II and expected number of visits of the system to some states of interest in a given interval of time have been obtained. A numerical approximation method is used to solve the system of integral equations. Such a transient state analysis provides an insight to the various characteristics of functioning of the system and is useful when it is desired to monitor the system over a finite horizon of time.

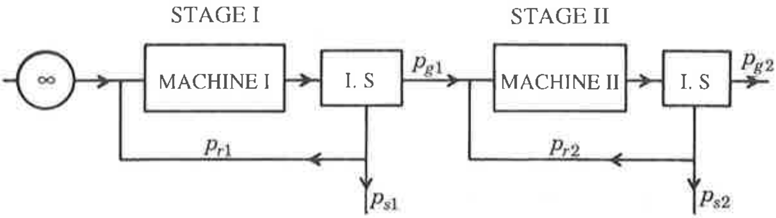
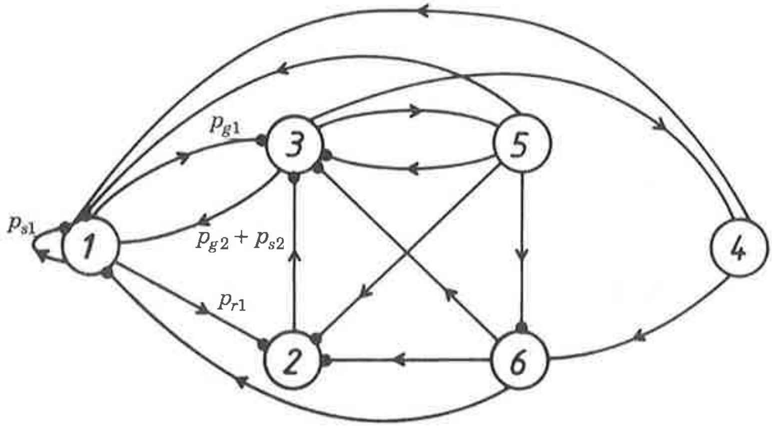


Figure 1. Schematic diagram of the production system



—●— indicates that the transition is regenerative

Figure 2. One-step transition diagram

Table 2: Effect of processing rate of M/c I on the expected output
$\lambda_2 = 2, \lambda_3 = 2, \lambda_4 = 1$

λ_1	t	Jobs by M/c I	Jobs by M/c II	Reworked jobs by M/c I
1	0.5	0.40542	0.03212	0.00345
	1.0	0.70618	0.15423	0.02054
	1.5	0.99099	0.36556	0.04848
	2.0	1.27554	0.63885	0.07965
	2.5	1.55947	0.95048	0.11085
2	0.5	0.66295	0.07175	0.00621
	1.0	1.04283	0.30715	0.03415
	1.5	1.38993	0.64977	0.07562
	2.0	1.72968	1.03976	0.11819
	2.5	2.06346	1.44813	0.15855
3	0.5	0.82858	0.10836	0.00844
	1.0	1.22511	0.41930	0.04339
	1.5	1.60278	0.82466	0.09131
	2.0	1.97717	1.25941	0.13750
	2.5	2.34574	1.70134	0.17992

Table 3: Effect of processing rate of M/c I on the expected output
$\lambda_2 = 2, \lambda_3 = 2, \lambda_4 = 1$

λ_1	t	Reworked jobs by M/c II	Blocked state	Rework state
1	0.5	0.02059	0.02566	0.05643
	1.0	0.06351	0.11048	0.11443
	1.5	0.11840	0.24871	0.17480
	2.0	0.18249	0.42093	0.23885
	2.5	0.25354	0.61200	0.30603
2	0.5	0.04160	0.05526	0.09879
	1.0	0.11321	0.23429	0.18263
	1.5	0.19350	0.49965	0.26059
	2.0	0.28109	0.80295	0.33797
	2.5	0.37406	1.11987	0.41574
3	0.5	0.05745	0.08256	0.12767
	1.0	0.13729	0.33317	0.21521
	1.5	0.21917	0.67152	0.29105
	2.0	0.30694	1.03534	0.36547
	2.5	0.39937	1.40320	0.44007

Table 4: Effect of processing rate of M/c II on the expected output
$\lambda_1 = 1, \lambda_3 = 2, \lambda_4 = 1$

λ_2	t	Jobs by M/c I	Jobs by M/c II	Reworked jobs by M/c I
	0.5	0.38091	0.03110	0.00347
	1.0	0.65387	0.17502	0.02106
3	1.5	0.91949	0.41770	0.05100
	2.0	1.18480	0.72149	0.08583
	2.5	1.44860	1.06176	0.12160
	0.5	0.36317	0.03759	0.00349
	1.0	0.61912	0.20530	0.02153
4	1.5	0.87006	0.47132	0.05296
	2.0	1.11986	0.79540	0.09011
	2.5	1.36807	1.15423	0.12850
	0.5	0.34700	0.04471	0.00350
	1.0	0.58800	0.22953	0.02193
5	1.5	0.82501	0.50923	0.05442
	2.0	1.06095	0.84463	0.09308
	2.5	1.29555	1.21380	0.13304

Table 5: Effect of processing rate of M/c II on the expected output
$\lambda_1 = 1, \lambda_3 = 2, \lambda_4 = 1$

λ_2	t	Reworked jobs by M/c II	Blocked state	Rework state
	0.5	0.01058	0.02103	0.04194
	1.0	0.03744	0.10823	0.07433
3	1.5	0.07745	0.24418	0.10954
	2.0	0.12801	0.40405	0.14959
	2.5	0.18614	0.57514	0.19371
	0.5	0.00836	0.02339	0.03489
	1.0	0.03269	0.11586	0.05668
4	1.5	0.07114	0.24673	0.08272
	2.0	0.12041	0.39315	0.11452
	2.5	0.17708	0.54642	0.15094
	0.5	0.00764	0.02633	0.02944
	1.0	0.03171	0.11976	0.04459
5	1.5	0.07032	0.24170	0.06549
	2.0	0.11963	0.37409	0.09282
	2.5	0.17606	0.51125	0.12508

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